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LETTER TO THE EDITOR

A self-consistent description of ruptures in an elastic medium: an application to earthquakes

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Abstract. We present a *general* self-consistent procedure using lattice Green's functions to find the stress redistribution due to multiple ruptures in a discrete elastic medium. We use this method to study a dipole version of the quasi-static crack-propagation model for earthquakes proposed by Chen *et al*. Unlike the earlier treatment we do not assume that ruptures occur independently. We obtain the stresses self-consistently and this changes the scaling behaviours of the frequency-size (seismic moment, energy) distribution. We make a careful comparison of our results with recent analysis of seismological data.

Earthquakes typically arise from faulting instabilities in the earth's crust which occur suddenly and release the stress accumulated due to slow movement of the tectonic plates. The instabilities include brittle fracture of rocks and slips along fractured surfaces which result in rich and complex dynamics. Of particular interest are the self-similar behaviours exhibited by earthquakes: the best known example is the Gutenberg–Richter law [2] which states that the number of earthquakes with energy greater than E, $n(E)$, follows a power law: $n(E) \propto E^{-B}$ (see later for a detailed discussion). Recently, in the context of 'self-organized criticality' [3], several models that capture some of the self-similar features of earthquakes have been studied [1, 5–10]. One class of models, originally suggested by Burridge and Knopoff [4], deals with the stick–slip behaviour at the faults [6–9]; while successful in elucidating some features of earthquakes, such models do not include the long-ranged stress redistribution that occurs following a slip event. We focus on a different class of *quasi-static* models that describe the entire seismic zone as an elastic medium and include some of the long-ranged stress redistribution but neglect short-time dynamics.

We model the seismic zone as a discretized elastic medium (a lattice of blocks) subjected to an externally imposed shear stress. Each block is randomly assigned a threshold stress which sets the maximum stress beyond which it will rupture. As the external stress increases blocks rupture initiating an earthquake that corresponds to a sequence of ruptures. After the earthquake the ruptures are assumed to heal, the process is repeated and statistics of earthquake properties can then be studied. Models in this class have been considered previously: the dipole model [1] in which forces in only one direction are included and a double-couple model [10] in which complete force and torque balance are imposed. The latter model amounts to a discretized version of the equations of continuum elasticity. For simplicity these studies treated each rupture independently of the other ruptures and did not constrain the stress across a rupture to be zero during the earthquake. We incorporate this essential feature in computing the stress redistribution during the earthquake.

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The main results of our work are as follows: (i) We have developed a *general* self-consistent procedure using lattice Green's functions to find the stress redistributions due to multiple ruptures. (ii) We illustrate the method by studying the dipole model numerically in 2 and 3 dimensions. We find that the self-consistent solution leads to new scaling behaviour for the frequency-size (seismic moment, energy) distributions of earthquakes different from that of the Chen, Bak, Obukhov model [1]. We finally review crossover effects from small to large earthquakes in seismological data [15] and show that our results for B are consistent with the data.

Dipole model. We model the seismic zone as a lattice of blocks subjected to a slowly increasing external shear stress in the xy plane along x where z is chosen along the vertical. Equilibrium elasticity conditions are imposed on the stresses defined at the centres of the blocks. In the 2D model the width of the crust (along z) is neglected. We restrict our attention to the shear mode of fracture and consider only shear ruptures along x in 2D and xz planes in 3D. Each block is randomly assigned a threshold stress between 0 and 1 which sets the maximum shear stress the block can sustain and beyond which it will rupture. We only consider forces along x and denote the (compressional) stress at the centre of a block r by $\sigma_x(r) \equiv \sigma_{xx}(r)$ and the shear stresses, e.g. in the xy plane by $\sigma_y(r) \equiv \sigma_{xy}(r)$. In the quasi-static approximation, the vanishing of the net force on each block implies that the stresses satisfy $\sum_i D_i \sigma_i = 0$; the discrete lattice gradient D_i is defined by $D_i f(r) \equiv [f(r+\hat{e}_i/2) - f(r-\hat{e}_i/2)]$, where $\hat{e}_i$ is the unit vector along $i = x$, y, or z.

The slow increase of the external shear stress is realized by adding a small stress to all σ_y until a block ruptures. This results in a long-ranged redistribution of the elastic forces that decay as $1/r^d$ with the distance r from the rupture. The resulting stress configuration in turn could trigger more blocks to rupture causing an avalanche in the system which we define as an earthquake. The stress redistribution can be considered instantaneous since it occurs with the speed of sound, much faster than the geological timescales involved in the build-up of stress. Therefore, we hold the external stress fixed during an earthquake. When the stresses in all the (unruptured) blocks are below their corresponding thresholds, the earthquake ends. After the earthquake, all the thresholds of the fractured surfaces are re-set to a random number between 0 and 1 signifying healing of the fractured blocks before the start of the next earthquake.

Self-consistent method. Below we briefly describe the basic formalism for the 2D dipole model; generalizations to the double-couple model and to 3D are straightforward. Consider a single rupture of a block at r_0. The corresponding shear stress σ_y decreases from its value σ_0 to zero resulting in a force imbalance and subsequent stress redistribution. The stress distributions before and after the rupture $\sigma_i^{\mathrm{old}}(r)$ and $\sigma_i^{\mathrm{new}}(r)$ differ by σ_2' the additional stress due to the rupture for $i = x, y$:

$$\sigma_i'(r) = \sigma_i^{\mathrm{new}}(r) - \sigma_i^{\mathrm{old}}(r). \tag{1}$$

Since, σ_i^{new} also satisfies force-balance, it follows that $\sum_{i=x,y} D_i \sigma_i' = 0$. Note that σ' is linearly related to the additional displacement u' via Hooke's law throughout the system except at the ruptured surface. It is convenient to separate out this violation of Hooke's law at the ruptured block:

$$\sigma_y' = \sigma_y^{\mathrm{el}} + \sigma^{\mathrm{ne}} \delta_{r,r_0} \qquad \sigma_x' = \sigma_x^{\mathrm{el}}. \tag{2}$$

In the preceding $\sigma_i^{\rm el}$ is the elastic part of the stress and $\sigma^{\rm ne}$ represents the non-elastic part of the stress drop proportional to the 'slip' at the ruptured surface. It follows that $\sigma^{\rm el}$ satisfies $\sum_{i=x,y} D_i\sigma_i^{\rm el}(\boldsymbol{r}) + F_y^{\rm dipole}(\boldsymbol{r}) = 0$, where the dipole force is given by

$$F_y^{\rm dipole}(\boldsymbol{r}) = f_y(\delta_{\boldsymbol{r},\boldsymbol{r}_0-\hat{e}_y/2} - \delta_{\boldsymbol{r},\boldsymbol{r}_0+\hat{e}_y/2}) \tag{3}$$

and $f_y = \sigma^{\rm ne}$. If one considers the full force and torque balance conditions [10], not just along x, one obtains an effective double-couple instead of a dipole. To solve for $\sigma^{\rm el}$, we need Hooke's law relating $\sigma^{\rm el}$ to the additional elastic displacement due to the ruptures. We assume the following form of Hooke's law: $\sigma_x^{\rm el} = D_x u_x'$ and $\sigma_y^{\rm el} = D_y u_x'$ where u_x' is the displacement along x (the same assumptions are made in the CBO model [1]). This permits us to determine $\sigma_{x,y}^{\rm el}$†. The result for $\sigma_y^{\rm el}$ (this is sufficient since we consider shear fractures only) is

$$\sigma_y^{\rm el}(\boldsymbol{r}) = -f_y G_y(\boldsymbol{r} - \boldsymbol{r}_0) \tag{4}$$

where the lattice Green's function G_y is given by

$$G_y(\boldsymbol{r}) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} {\rm d}k_x\,{\rm d}k_y\,\frac{{\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{r}}(1-\cos k_y)}{2-(\cos k_x + \cos k_y)}. \tag{5}$$

The condition that the stress at the ruptured surface is zero determines the 'magnitude' of the dipole force $f_y : f_y = -2\sigma_0$.

Now we consider the stress redistribution due to many ruptures simultaneously. In analogy with the single rupture case, the additional stress caused by the many ruptures can be viewed as being due to dipole forces at each of the ruptured blocks. Suppose there are n_y ruptured blocks at $\{\boldsymbol{r}_{0i},\ i = 1,\ldots,n_y\}$; the elastic part of the extra stress due to these ruptures can be expressed as (cf (4))

$$\sigma_y^{\rm el}(\boldsymbol{r}) = -\sum_{i=1}^{n_y} f_{iy} G_y(\boldsymbol{r} - \boldsymbol{r}_{0i}). \tag{6}$$

The magnitude of the dipole forces can be obtained self-consistently by solving a set of linear equations corresponding to the boundary conditions of zero stress at the ruptured surfaces:

$$\sigma_y^{\rm new}(\boldsymbol{r}_{0i}) = \sigma_y^{\rm old}(\boldsymbol{r}_{0i}) + \sigma_y^{\rm el}(\boldsymbol{r}_{0i}) + f_{iy} = 0 \tag{7}$$

for $i = 1,\ldots,n_y$. The new stress distribution can then be obtained from the dipole forces $\{f_{iy}\}$ by using (1) and (6).

Results for the dipole model. As the stress is increased, earthquakes occur and the ruptures are healed, the system evolves to a *critical* state. We characterize each earthquake by quantities analogous to the energy release E and seismic moment magnitude M_0 for real earthquakes. Since, the dipole force f_{iy} is the body force equivalent [11] of the slip across the broken block i, it is a measure of the local slip. The stress drop at a broken block

† We assume that the redistribution of elastic forces occurs over an infinite medium and solve the equations for an infinite system. For our finite system this corresponds to an open boundary condition with the activity outside the system being neglected.

is simply σ^{old} since $\sigma^{\text{new}} = 0$. The energy released in an earthquake E is taken to be the elastic work done by the stress drop to create a mean slip integrated over the seismic zone [12], and therefore, we define $E \approx -\sum_i f_i \cdot \sigma^{\text{old}}(i)$ where the sum is over all the ruptured blocks. Similarly, since the magnitude of the seismic moment in the xy plane is proportional to the total mean slip along x, we define $M_0 \approx |\sum_i f_{iy}|$ where the sum is over all the broken blocks. We also monitor the number N of the broken blocks during an earthquake.

We display in figure 1 typical spatial patterns (broken block configurations) of large earthquakes that occur in 2D and 3D. The pattern of a 2D earthquake (figure 1(*a*)) is that of a linear crack with some scatter due to the rupture of weak spots caused by long-ranged interactions. For 3D, the ruptures are confined to adjacent xz planes with some scatter as shown in figure 1(*b*) and have the form of a planar crack. A more quantitative analysis can be made as follows: We can characterize 2D earthquakes by a length L defined as the root mean square deviation of the x coordinates of the ruptured blocks. Our data are consistent with the scaling of the energy $E \approx L^2$ (see below). Note that for a linear crack of size L we expect the surface 'area' of broken surfaces and the mean displacement in the ruptured blocks to scale as L leading to $E \propto L^2$. We compute numerically the energy release due to the creation of a linear crack of length L and verify this scaling for $L \geqslant 10$. Thus 2D earthquakes in our model correspond to a 'macroscopic' linear rupture.

Next we discuss the distributions of the energy release (E) in both 2D and 3D. The results are obtained using a system of size 80×80 at $d = 2$ and $20 \times 20 \times 20$ at $d = 3$ for 100 000 earthquakes after skipping the first 20 000 earthquakes to evolve the system to a critical state. In figure 2 we show a log–log plot of the distribution $P(E)$ versus E; the linear portion which extends up to $\tilde{E}(L)$ corresponds to a power law $P(E) \approx E^{-1-B}$ with $B = 0.8 \pm 0.1$ for 2D and $B = 1.0 \pm 0.1$ for 3D. In contrast, the CBO model [1] yields $B \approx 0.35$ for 2D and $B \approx 0.6$ for 3D. Note that the value for B in 3D is in agreement with the mean field exponent derived in [5] using a model with gap-dynamics. Beyond $\tilde{E}$ the distribution has a hump which is a *finite-size* effect. We find that $\tilde{E}$ increases with L and consequently the probability in the hump scales to zero with increasing L. Such humps also occur in the distributions of other quantities.

The distribution for the seismic moment M_0 (not shown here) scales similarly with the same exponents as that of the energy. This is expected since the average slip due to rupture–rupture interactions is proportional to the linear size of the macroscopic rupture. However, the distribution $P(N)$ for the number of fractured blocks N follows a different power law as shown in figure 3. We find $P(N) \propto N^{-1-B_1}$, with $B_1 = 1.1 \pm 0.1$ in 2D and $B_1 = 1.3 \pm 0.1$ for 3D models†. Thus the self-consistency of the stress redistribution leads to $B \neq B_1$ in contrast to the CBO model [1] for which E and N have identical scaling behaviours.

We now compare our results with seismological data and discuss crossover effects. The original form of the Gutenberg–Richter law states that the rate of earthquakes with (surface wave) magnitude larger than M_s, $n(M_s)$ scales as $\log_{10}[n(M_s)] = A - bM_s$ with an exponent b. In terms of the total energy release E, the rate of earthquakes with energy larger than E, $n(E)$, scales as $n(E) = aE^{-B}$ with an exponent B. It is generally accepted that M_s is proportional to the logarithm of E: $\log_{10}(E) = d + cM_s$; this yields $b = cB$.

† We also find that modifications of our model, such as (i) the inclusion of non-shear mode of ruptures due to the tensile stress σ_{xx} exceeding its threshold in $d = 2$ and (ii) choosing the threshold distribution to be uniform between p_0 and 1 with $p_0 = 0.1, 0.2$, leave the exponents B, B_1 invariant while changing non-universal features such as the shape of the hump.

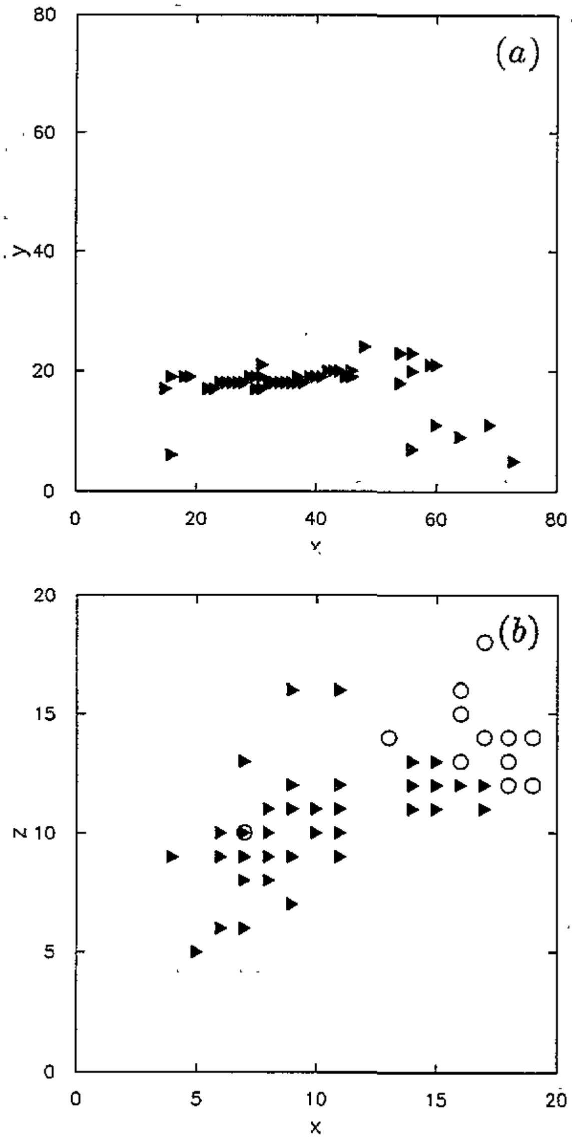
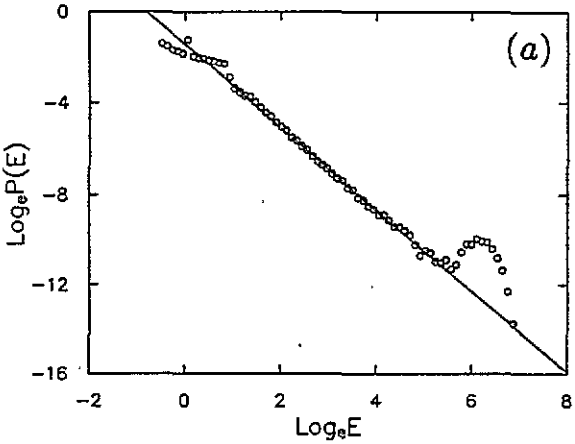


Figure 1. Spatial patterns of typical large earthquakes in 2D and 3D for the dipole model: (*a*) The triangles represent the ruptured blocks in a 2D earthquake in a 80×80 system. (*b*) The ruptured blocks in two adjacent layers (x–z planes) of the 3D system are shown (in this particular earthquake, only blocks in these two layers rupture). The triangles and open circles denote the locations of the ruptured blocks in x–z planes at $y = 16$ and $y = 17$ respectively in a $20\times20\times20$ system.

There are two possible reasons for a crossover in the scaling behaviours from large to small earthquakes: (i) M_s calculated from seismic-wave amplitudes at a period of 20 seconds underestimates the energy of large earthquakes of longer duration. Recent analysis of data [13] indicates a crossover from $c \approx 3/2$ for $M_s > 6.8$ to $c \approx 1$ for $M_s < 5.3$ in agreement with earlier theoretical analysis [14]. (ii) The second reason is geometrical. While the



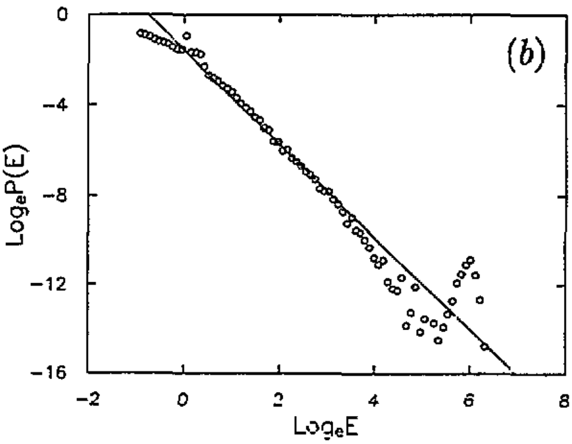


Figure 2. Log–log plot of the probability distribution of the energy release $P(E)$ versus E: (*a*) for the 2D model (straight line has a slope ≈ 1.8); (*b*) for the 3D model (straight line has a slope ≈ 2.0).

extent of small earthquakes is not constrained the down-dip width of a large earthquake is limited by the thickness of the seismogenic layer. Therefore, large earthquakes correspond to $d = 2$ and small ones to $d = 3$. Recent analysis of the frequency-size relation [15] reveals a break in the self-similar behaviour at a magnitude around 6.5 to 7.5 due to the change in effective dimensionality. This leads to a change in b from roughly 1 for small earthquakes to 3/2 for large ones.

It is reasonable to view earthquakes with M_s between 2.0 and 5.5, as 3D earthquakes, and hence, use $c \approx 1$ and deduce a value of $B \approx 1$. The situation for determining B for 2D earthquakes, on the other hand, is far less clear due to the two crossovers involved. If we assume that earthquakes with $M_s > 7.5$ are two-dimensional and use $c \approx 3/2$, we again obtain $B \approx 1$. With these caveats, the B values obtained for our model in $d = 2$ and 3 are in rough agreement with the above B values for large and small earthquakes respectively.

We conclude by pointing out that the self-consistent method proposed here can be used to study ruptures in the more realistic double couple model [16] and also models that include pre-existing fault structure.

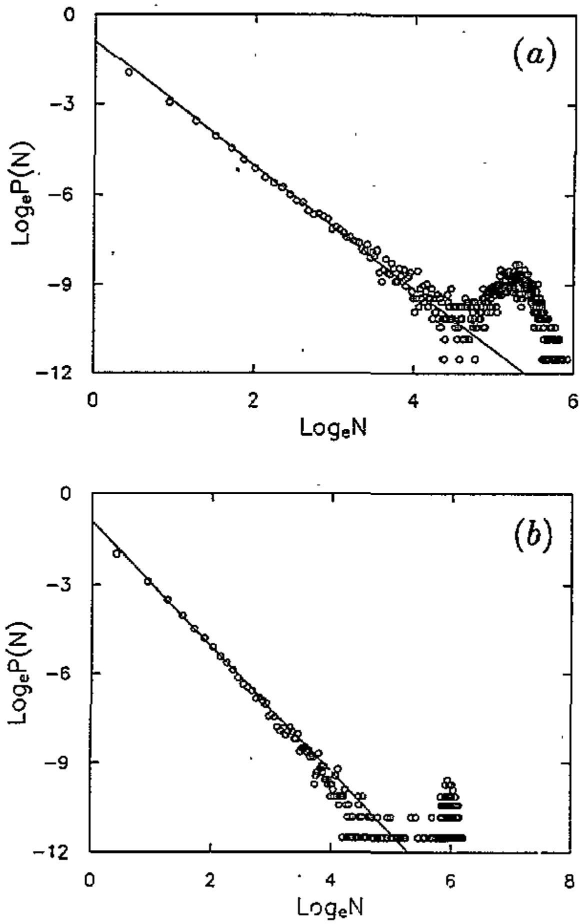


Figure 3. Log–log plot of the probability distribution of the number of broken blocks $P(N)$ versus N: (*a*) for the 2D model (straight line has a slope ≈ 2.1); (*b*) for the 3D model (straight line has a slope ≈ 2.3).

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References

[1] Chen K, Bak P and Obukhov S P 1991 *Phys. Rev.* A **43** 625
[2] Gutenberg B and Richter C F 1956 *Ann. di Geofis.* **9** 1
[3] Bak P, Tang C and Wiesenfeld K 1987 *Phys. Rev. Lett.* **59** 381
[4] Burridge R and Knopoff L 1967 *Bull. Seismol. Soc. Am.* **57** 341
[5] Sornette A and Sornette D 1989 *Europhys. Lett.* **9** 192
[6] Carlson J M and Langer J S 1989 *Phys. Rev. Lett.* **62** 2632; 1989 *Phys. Rev.* A **40** 2803
[7] Bak P and Tang C 1989 *J. Geophys. Res.* B **94** 15635
[8] Ito K and Matsuzaki M 1990 *J. Geophys. Res.* B **95** 6853

[9] Bak P and Chen K 1991 *Fractals and Their Application to Geology* ed C Barton and P LaPointe (Denver: Geological Society of America) and references therein
[10] Xu H-J, Bergersen B and Chen K 1992 *J. Phys. A: Math. Gen.* 25 L1251; 1993 *J. Phys. (France)* 3 2029
[11] Aki K and Richards P G 1980 *Quantitative Seismology, Theory and Methods* (San Francisco: Freeman)
[12] Scholz C H 1990 *The Mechanics of Earthquakes and Faulting* (Cambridge: Cambridge University Press)
[13] Ekstrom G and Dziewonski A M 1988 *Nature* 332 319
[14] Kanamori H and Anderson D 1975 *Bull. Seismol. Soc. Am.* 65 1073
[15] Pacheco J F, Scholz C H and Sykes L R 1992 *Nature* 355 71
[16] Bhagavatula R, Chen K, Jayaprakash C and Xu H-J unpublished